K.S Probability (Theory)

Let, a random experiment has a outcome of variable x takes the values $x_1, x_2, x_3 \cdots x_n$, with probability $P_1, P_2, P_3 \cdots P_n$ respectively, then the mathematical expectation of variable x is denoted by $E(x)$ which is denoted by,

$$E(x) = x_1 P_1 + x_2 P_2 + \cdots + x_n P_n$$

$$= \sum_{i=1}^{n} x_i P_i \quad \text{and} \quad \sum_{i=1}^{n} P_i = 1$$

Example : Let, two coins are tossed simultaneously and x be the no. of head.

So, x take value 0, 1, 2

Sample space $= \{HH, HT, TH, TT\}$

$2^n = 2^2 = 4$

probability that $x = 0$

$$P_1 = \frac{1}{4}$$

probability that $x = 1$,

$$P_2 = \frac{1}{2} \quad \left(\frac{2}{4}\right)$$

probability that $x = 2$

$$P_2 = \frac{1}{4}$$

NOTE
Expectation value of x means the average value / mean value of x

Mathematical expectation $E(x) = P_1 x_1 + P_2 x_2 + P_3 x_3$

$$= 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$$

Example find the expectation value of number of tells when 3 Co are tossed ?

$\Rightarrow$ Sample space $= \{HHH, HHT, HTT, TTT, TTH, THH, HT$
$THT\}$

probability that $x = 0$, $P_1 = \frac{1}{8}$

" that $x = 1$, $P_2 = \frac{3}{8}$

$x = 2$ $P_3 = \frac{3}{8}$

$x = 3$, $P_4 = \frac{1}{8}$

Mathematical expectation $E(x) = P_1 x_1 + P_2 x_2 + P_3 x_3 + P_4 x_4$

$$= 0 + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$$

$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8}$$

A box contains 4 white and 6 black balls. 3 balls are drawn, then find the expectation of the number of white balls.

Soln: Here, x be the number of white ball.

x take the value 0, 1, 2, 3

probability that $x = 0$

$$P(x=0) = \frac{{}^4C_0 \; {}^6C_3}{{}^{10}C_3} = \frac{1 \times \frac{6\times5\times4}{3\times2\times1}}{\frac{10\times9\times8}{3\times2\times1}}$$

($^nC_0 = 1$)

$$= \frac{6\times5\times4}{10\times9\times8} = \frac{1}{6}$$

probability that $x = 1$,

$$P(x=1) = \frac{{}^4C_1 \; {}^6C_2}{{}^{10}C_3} = \frac{4 \times \frac{6\times5}{2\times1}}{\frac{10\times9\times8}{3\times2}}$$

$$= \frac{2\times6\times5\times3\times2}{10\times9\times8} = \frac{1}{2}$$

probability that $x = 2$

$$P(x=2) = \frac{{}^4C_2 \cdot {}^6C_1}{{}^{10}C_3}$$

$$= \frac{\frac{4\times3}{2}\times6}{\frac{10\times9\times8}{6}} = \frac{12\times6\times6\times3}{2\times10\times9\times8}$$

$$= \frac{3}{10}$$

probability that $x = 3$,

$$P(x=3) = \frac{{}^4C_3 \cdot {}^6C_0}{{}^{10}C_3} = \frac{1 \times \frac{4\times3\times2}{3\times2\times1}}{\frac{10\times9\times8}{3\times2}}$$

$$= \frac{1}{30} = \frac{4\times3\times2}{10\times9\times8}$$

Expectation value $E(x) = p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4$

$$= \frac{1}{2} + 2\times\frac{3}{10} + 3\times\frac{1}{30}$$

$$= \frac{5+6+1}{10} = \frac{12}{10} = \frac{6}{5}$$

① 3 defective bulbs are mixed with 7 good bulbs, let 'x' be the number of defective bulbs when 3 bulbs are drawn at randomly find the mean of x or expectation of x ?

x be the no. of defective bulbs,

$x = 0, 1, 2, 3,$

probability that $x = 0$,

$$P(x=0) = \frac{{}^3C_0 \; {}^7C_3}{{}^{10}C_3}$$

$$= \frac{1 \times \frac{7\times6\times5}{3\times2\times1}}{\frac{10\times9\times8}{3\times2\times1}} = \frac{7\times6\times5}{10\times9\times8} = \frac{7}{24}$$

probability that $x = 1$, $P(x=1) = \frac{{}^3C_1 \; {}^7C_2}{{}^{10}C_3}$

probability that $x = 2$,

$$P(x=2) = \frac{{}^3C_2 \; {}^7C_1}{{}^{10}C_3}$$

$$= \frac{\frac{3\times2}{2} \times 7 \times 3}{10\times9\times8}$$

$$= \frac{3\times7\times3}{10\times9\times4} = \frac{7}{40}$$

$$= \frac{3 \times \frac{7\times6}{2}}{\frac{10\times9\times8}{3\times2}}$$

$$= \frac{7\times3\times3}{10\times4\times4}$$

$$= \frac{21}{40}$$

$$P(x=3) = \frac{{}^3C_3 \cdot {}^7C_0}{{}^{10}C_3}$$

$$= \frac{\frac{3\times2\times1}{3\times2\times1} \cdot 1 \times 3}{\frac{10\times9\times8}{3\times2}}$$

$$= \frac{1}{120}$$

Expectation value of $E(x) = 1 \times \frac{21}{40} + 2 \times \frac{7}{40} + 3 \times \frac{1}{120}$

$$= \frac{63 + 42 + 3}{120} = \frac{108}{120} = \frac{54}{60} = \frac{27}{30}$$

$$= \frac{9}{10}$$

A single dice = 6

Two dice = 6×6
$= 36$

$S = \{(x, y) : x, y \in \{1, 2, 3, 4, 5, 6\}\}$

$$S = \begin{Bmatrix} (1,1),(1,2),(1,3) \ldots\ldots (1,6), \\ (2,1),(2,2),(2,3) \ldots\ldots (2,6), \\ (3,1),(3,2),(3,3) \ldots (3,6), \\ (4,1),(4,2),(4,3) \ldots (4,6), \\ (5,1),(5,2),(5,3) \ldots (5,6), \\ (6,1),(6,2),(6,3) \ldots (6,6). \end{Bmatrix}$$

* A Coin is tossed 4 times,

$2^4 = 16$

$S = \{HHHH, HHHT, HHTH, HTHH, HTHT, HTTH, HTTT$
$THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH,$
$TTTT\}$

* A Coin and A dice is tosed
is throwm,

$S = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),$
$(T,1),(T,2),(T,3),(T,4),(T,5),(T,6),\}$

* ~~draft~~

* A Coin is tosed and then a dice is rolled only incase a head is shown on the coin. (tell আগাগোড়া কিন্তু বলতে লাগবে না)

$\Rightarrow S = \{T, (H,1),(H,2),(H,3),(H,4),(H,5),(H,6)\}$

* Two Boys and two girls are room χ and one boys and three girls are room γ, is specified the sample is space, the experiment in which a room is selected and a person for

let, in the room χ, B_1, B_2 (Boys)
G_1, G_2 (girls)

in the rooom γ, B_3
G_3, G_4, G_5

$S = \{(\chi, B_1),(\chi, B_2),(\chi, G_1),(\chi, G_2),(\gamma, B_3),$
$(\gamma, G_3),(\gamma, G_4),(\gamma, G_5)\}$

Binomial distribution.

A discrete random variable x is said to be have a Binomial distribution with parameter 'n' and 'p' if the probability mass function (pmf) is given by —

$$f(x) = P(x = x) = {}^{n}C_{x} \cdot p^{x} q^{n-x} \quad ; \quad \text{where } x = 0, 1, 2, 3 \cdots, n \text{ and } p+q = 1$$

Binomial distribution also written as,

$$B(n, p) = {}^{n}C_{x} \cdot p^{x} q^{n-x}$$

where, n = number of trials
x = " " success
p = probability of success
q = " " failure

find the mean, variance, standard daviation of binomial distribution.

① Mean, $E(x) = \sum_{x=0}^{n} x \cdot f(x)$

$$= \sum_{x=0}^{n} x \cdot {}^{n}C_{x} \cdot p^{x} q^{n-x}$$

$$= 0 + {}^{n}C_{1}\, p \cdot q^{n-1} + 2 \cdot {}^{n}C_{2}\, p^{2} \cdot q^{n-2} + 3\, {}^{n}C_{3}\, p^{3} q^{n-3} + \cdots + n \cdot {}^{n}C_{n} \cdot p^{n} \cdot q^{n-n}$$

$$= npq^{n-1} + 2 \cdot \frac{n!}{2!\,(n-2)!} p^{2} \cdot q^{n-2} + 3 \cdot \frac{n!}{3!\,(n-3)!} p^{3} \cdot q^{n-3} + \cdots + n\, p^{n}$$

$$= npq^{n-1} + \frac{n!}{(n-2)!} p^{2} \cdot q^{n-2} + \frac{n!}{2\,(n-3)!} p^{3} \cdot q^{n-3} + \cdots + np^{n}$$

$$= np\left[q^{n-1} + (n-1) \cdot p\, q^{n-2} + \frac{(n-1)(n-2)}{2} p^{3} \cdot q^{n-3} + \cdots + p^{n-1}\right]$$

$$= np\left[{}^{n-1}C_{0} \cdot q^{n-1} + {}^{n-1}C_{1} \cdot p \cdot q^{n-2} + {}^{n-1}C_{2} \cdot p^{3} q^{n-3} + \cdots + {}^{n-1}C_{n-1}\, p^{n-1}\right]$$

$$= np\,[q + p]^{n-1} \qquad [p+q = 1]$$

$$= np\,(1)^{n-1}$$

$$= np$$

So, mean of binomial distribution is np.

⑥ Varience of Binomial Distribution

$$E(x^2) = E\{x(x-1)+x\}$$
$$= E\{x(x-1)\} + E(x)$$
$$= E\{x(x-1)\} + np$$

Now, $E\{x(x-1)\}$

$$= \sum_{x=0}^{n} \{x(x-1)\} f(x) = \sum_{x=0}^{n} \{x(x-1)\} {}^{n}C_x \cdot p^x \cdot q^{n-x}$$

$$= 0 + {}^{n}C_1 \cdot p^1 \cdot q^{n-1} \cdot 0 + 2\, {}^{n}C_2 \cdot p^2 q^{n-2} + 6\, {}^{n}C_3\, p^3 q^{n-3}$$
$$+ \cdots + n(n-1)\, {}^{n}C_n\, p^n$$

$$= 2\frac{n!}{2!(n-2)!} p^2 q^{n-2} + 6\frac{n!}{3!(n-3)!} p^3 q^{n-3} + \cdots + n(n-1)p^n$$

$$= \frac{n!}{(n-2)!} p^2 q^{n-2} + \frac{n!}{(n-3)!} p^3 q^{n-3} + \cdots + n(n-1)p^n$$

$$= n(n-1)\, p^2 \left\{ q^{n-2} + (n-2)\, p\, q^{n-3} + \cdots + p^{n-2} \right\}$$

$$= n(n-1)\, p^2 \, (p+q)^{n-2}$$

$$= n(n-1)\, p^2$$

$$E(x^2) = n(n-1)\, p^2 + np$$
$$= n^2 p^2 - np^2 + np$$
$$= np\,(np + 1 - p)$$

$$\text{Varience} = E(x^2) - \{E(x)\}^2$$

$$= np\,(np+1-p) - n^2 p^2$$

$$= np - np^2 \qquad p + q = 1$$

$$= np(1-p) \qquad q = 1 - p$$

$$= npq$$

$$S.D = \sqrt{\text{varience}}$$

$$S.D = \sqrt{npq}$$

● Show that, Mean of Binomial distribution is greater than equal to Varience.

⇒ We know that, mean and varience of Binomial distribution is np and npq respectively.

where, $p+q=1$

$\Rightarrow q \leq 1$

$\Rightarrow npq \leq np$

Varience of B.D $\leq$ Mean of B.D

Q.2 find the mode of B.D.

We know that, mode is the value of x for which distribution mass function is maximum.

probability mass function (p.m.f) of Binomial distribution,

$$P(X=x) = {}^{n}C_{x} \cdot p^{x} \cdot q^{n-x}$$

Let, $X = x_1$ Be the mode of the Binomial distribution.

proof of $X = x_1 - 1$

$$\frac{f(X=x_1-1) \leq f(X=x_1) \geq f(X=x_1+1)}{f(X=x_1) \geq f(X=x_1-1)}$$

$$\Rightarrow {}^{n}C_{x_1} \cdot p^{x_1} \cdot q^{n-x_1} \geq {}^{n}C_{x_1-1}\, p^{x_1-1}\, q^{n-x_1+1}$$

$$\Rightarrow \frac{n!}{x_1!\,(n-x_1)!} \cdot p^{x_1} \cdot q^{n-x_1} \geq \frac{n!}{(x_1-1)!\,(n-x_1+1)!} p^{x_1-1} \cdot q^{n-x_1+1}$$

$$\Rightarrow \frac{1!}{x_1\,(x_1-1)!\,(n-x_1)!} \geq \frac{1!}{(x_1-1)!\,(n-x_1+1)(n-x_1)!} p^{-1} q$$

$$\Rightarrow \frac{p}{x_1} \geq \frac{q}{(n-x_1+1)}$$

$$\Rightarrow p(n-x_1+1) \geq qx_1$$

$$\Rightarrow pn - px_1 + p \geq qx_1$$

$$\Rightarrow np + p \geq qx_1 + px_1$$

$$\Rightarrow p(n+1) \geq x_1(p+q)$$

$$p(n+1) \geq x_1 \quad \text{——— (1)}$$

$f(x = x_1) \geq f(x = x_1 + 1)$

$\Rightarrow {}^{n}C_{x_1} \cdot p^{x_1} \cdot q^{n-x_1} \geq {}^{n}C_{x_1+1} \cdot p^{x_1+1} \cdot q^{n-x_1-1}$

$\Rightarrow \frac{n!}{x_1!(n-x_1)!} p^{x_1} q^{n-x_1} \geq \frac{n!}{(x_1+1)!(n-x_1-1)!} p^{x_1+1} \cdot q^{n-x_1-1}$

$\Rightarrow \frac{n!}{x_1(x_1-1)!(n-x_1)(n-x_1-1)!} p^{x_1} q^{n-x_1} \geq \frac{n!}{(x_1+1)x_1(x_1-1)!(n-x_1)(n-x_1-1)!} p^{x_1+1} q^{n-x_1-1}$

$\Rightarrow \frac{1}{n-x_1} \geq \frac{p \cdot q^{-1}}{x_1+1}$

$\Rightarrow q(x_1+1) \geq p(n-x_1)$

$\Rightarrow qx_1 + q \geq np - px_1$

$\Rightarrow x_1 \geq np - q \qquad p+q=1$

$q = 1-p$

$x_1 \geq np - 1 + p$

$x_1 \geq p(n+1) - 1 \qquad \text{---------- (II)}$

from (I) and (II)

$(n+1)p - 1 \leq x_1 \geq p(n+1)$

Case-1 when $p(n+1)$ is an integer,

$(n+1)p - 1$ also an integer

As x_1 is an integer at above condition, mode of binomial distribution are $(n+1)p - 1$ and $p(n+1)$.

Case-2 If $(n+1)p$ is not an integer,

$(n+1)p - 1$ " also not an integer,

But x_1 should be integer.

So, x_1 = integer in between $(n+1)p$ and $(n+1)p - 1$

Q.8 ** find the recurrence relation of Binomial distribution,

$$\mu_{r+1} = pq\left[nr\mu_{r-1} + \frac{d}{dp}(\mu_r) \right]$$

probability mass function (P.M.F)

⇒ The proof of Binomial distribution,

$$P(X = x) = {}^{n}C_x \cdot p^x \cdot q^{n-x}$$

μ_r define as,

$$\mu_r = E(x - \bar{x})^r \qquad [\bar{x} = np$$

$$= E\{x - np\}^r$$

$$\mu_r = \sum_{x=0}^{n} (x - np)^r \cdot f(x)$$

$$\Rightarrow \mu_r = \sum_{x=0}^{n} (x - np)^r \cdot {}^{n}C_x \, p^x \, q^{n-x} \quad \text{——— (1)}$$

Taking derivative of eqn (1) w.r.t to P, $[p+q=1, q=1-p$

$$\frac{d}{dp}(\mu_r) = \frac{d}{dp} \sum_{x=0}^{n} (x-np)^r \cdot {}^{n}C_x \, p^x (1-p)^{n-x}$$

$$\Rightarrow \frac{d}{dp}(\mu_r) = \sum_{x=0}^{n} \Big[r(x-np)^{r-1}(-n) \, {}^{n}C_x \, p^x (1-p)^{n-x} + (x-np)^r \, {}^{n}C_x \, x p^{x-1} (1-p)^{n-x} + (x-np)^r \, {}^{n}C_x \, p^x (n-x)(1-p)^{n-x-1}(-1) \Big]$$

$$\Rightarrow \frac{d}{dp}(\mu_r) = -nr \sum_{x=0}^{n} (x-np)^{r-1} \, {}^{n}C_x \, p^x (1-p)^{n-x} + \sum_{x=0}^{n} (x-np)^r \, {}^{n}C_x \, x p^{x-1} (1-p)^{n-x} + \sum_{x=0}^{n} (x-np)^r \, {}^{n}C_x \, p^x \cdot q^{n-x} \left[\frac{x}{p} - \frac{n-x}{q} \right]$$

$$= -nr\,\mu_{r-1} + \sum_{x=0}^{n} {}^{n}C_x (x-np)^r \, p^x q^{n-x} \left\{ \frac{xq - np + xp}{pq} \right\}$$

$$= -nr\,\mu_{r-1} + \sum_{x=0}^{n} {}^{n}C_x (x-np)^r \, p^x \cdot q^{n-x} \, \frac{[x(p+q) - np]}{pq}$$

$$= -nr\,\mu_{r-1} + \sum_{x=0}^{n} \frac{{}^{n}C_x (x-np)^r \, p^x q^{n-x} (x-np)}{pq}$$

$$\left[{}^{n}C_x (x-np)^{r-1} p^x q^{n-x} = \mu_{r-1} \right.$$

$$= -n\mu_{r-1} + \sum_{x=0}^{n} \frac{{}^{n}C_x (x-np)^{r+1} p^x q^{n-x}}{pq}$$

$$= -n\mu_{r-1} + \frac{1}{pq}\mu_{r+1}$$

$$\frac{d}{dp}(\mu_r) + n\mu_{r-1} = \frac{1}{pq}\mu_{r+1}$$

$$\mu_{r+1} = pq\left[n\mu_{r-1} + \frac{d}{dp}(\mu_r)\right]$$

Pol ① If the sum of mean and varience of B.D for 5 trials is 1.8. find the distribution.

② find the B.D for which the mean and varience 12 and 3 respectively.

Soln

The mean of the variation = np

Varience " " " = npq

A.T.P, $np + npq = 1.8$ $\quad p+q=1$

$np(1+q) = 1.8$

$5p(1+1-p) = 1.8$

$p(2-p) = \frac{1.8}{5}$

$5(2p - p^2) = 1.8$

$10p - 5p^2 - 1.8 = 0$

$5p^2 - 10p + 1.8 = 0$

$50p^2 - 100p + 18 = 0$

$25p^2 - 50p + 9 = 0$

$25p^2 - 45p - 5p + 9 = 0$

$5p(5p-9) - 1(5p-9) = 0$

$5p - 1 = 0$ | $p = 9/5$

$p = 1/5$ | $p = 1.8$

$= 0.2$

$5 \mid 225$
$5 \mid 45$
9

$p + q = 1$

$q = 1 - p$

$q = 0.8$

The distribution is $= {}^{5}C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{5-x}$ Ans

② $np = 12$

$n = \frac{12}{3} \times 4$

$= 16$

varience, $npq = 3$

$q = \frac{3}{12}$

$q = \frac{1}{4}$

$p + q = 1$

$p = 1 - \frac{1}{4} = \frac{3}{4}$

∴ The distribution ${}^{16}C_x \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)$

② poisson's distribution :

Let, x Be a discrete random variable which taken the value $0, 1, 2, 3 \ldots$ So has, Pmf (probability Mass function) $f(x)$ is given by —

$$f(x) = P(x = x) = \frac{e^{-m} \cdot m^x}{x!} \quad ; m > 0$$

Then x is called poisson variable and the distribution is called poisson's distribution.

$$x \sim P(m)$$

① Find the mean, variance and S.D of poisson's distribution.

probability mass function of poisson's distribution,

$$f(x) = \frac{e^{-m} \cdot m^x}{x!}$$

Now, mean; $\bar{x} = E(x) = \sum_{x=0}^{n} x \cdot f(x)$

$$= \sum_{x=0}^{n} x \frac{e^{-m} \cdot m^x}{x!}$$

$$= \frac{e^{-m} m}{1!} + \frac{2 e^{-m} m^2}{2!} + \frac{3 e^{-m} \cdot m^3}{3!} + \cdots\cdots\cdots$$

$$= m e^{-m} \left[1 + \frac{2m}{2!} + \frac{3m^2}{3 \times 2} + \cdots\cdots \right]$$

$$= m e^{-m} \left(1 + m + \frac{m^2}{2} + \cdots\cdots \right)$$

$$= m e^{-m} e^{m}$$

$$= m$$

Mean of poisson's distribution is $= m$

Varience, $E(x^2) = E\{x(x-1) + x\}$

$$= E\{x(x-1)\} + E(x)$$

$$= E\{x(x-1)\} + m$$

Now, $E\{x(x-1)\} = \sum_{x=0}^{n} x(x-1) \frac{e^{-m} \cdot m^x}{x!}$

$= 2 \frac{e^{-m} m^2}{2!} + 3 \cdot 2 \frac{e^{-m} m^3}{3!} + 12 \frac{e^{-m} m^4}{4!} + \cdots$

$= e^{-m} \cdot m^2 + e^{-m} m^3 + e^{-m} \frac{m^4}{2} + \cdots$

$= e^{-m} m^2 \left[1 + m + \frac{m^2}{2!} + \cdots\right]$

$= e^{-m} m^2 e^m$

$= m^2$ $\quad \therefore E\{x(x-1)\} + m = m^2 + m$

Varience $= E(x^2) - \{E(x)\}^2$

$= m^2 + m - m^2$ [In Case of poisson's ~~bari~~ distribution Mean = Varience]

$= m$

$S.D = \sqrt{\text{Varience}}$

$= \sqrt{m}$

* Show that mean & varience of P.D is equal.

Q find the mode of poisson's Distribution.

Ans The Pmf of poisson's ,, is,

$$f(x) = \frac{e^{-m} \cdot m^x}{x!}$$

Let, $x = x_1$ is mode of the P.D

$\therefore f(x_1)$ is the maximum value.

$$f(x_1 - 1) \leq f(x_1) \geq f(x_1 + 1) \quad \text{---(1)}$$

NOW, $f(x_1) \geq f(x_1 - 1)$

$\Rightarrow \frac{e^{-m} \cdot m^{x_1}}{x_1!} \geq \frac{e^{-m} m^{x_1 - 1}}{(x_1 - 1)!}$

$\Rightarrow \frac{1}{x_1 (x_1 - 1)!} \geq \frac{m^{-1}}{(x_1 - 1)!}$

$\frac{1}{x_1} \geq m^{-1}$

$\frac{m}{x_1} \geq 1$

$m \geq x_1$

$x_1 \leq m$ ——— (II)

NOW, $f(x_1) \geq f(x_1+1)$

$$\frac{e^{-m} \cdot m^{x_1}}{x_1!} \geq \frac{e^{-m} \cdot m^{x_1+1}}{(x_1+1)!}$$

$$\frac{1}{x_1!} \geq \frac{m}{(x_1+1)\, x_1!}$$

$(x_1+1) \geq m$

$x_1 \geq m-1$ ——— (III)

from equn (I)

$m-1 \leq x_1 \geq m$

Case-1 If m is an integer then $(m-1)$ is also an integer.

Ad, The point of x_1 is always integer at that Condn, mode of P.D is m and $m-1$

Case-2 If m is not an integer

then $(m-1)$ " " " "

But x_1 should be integer,

mode of P.D is the " Between m and $(m-1)$.

1 Prove that for the P.D, $\frac{f(x+1)}{f(x)} = \frac{m}{x+1}$

$$\frac{f(x+1)}{f(x)} = \frac{\frac{e^{-m}\, m^{x+1}}{(x+1)!}}{\frac{e^{-m}\, m^{x}}{x!}} = \frac{e^{-m}\, m^{x+1}}{(x+1)!} \cdot \frac{x!}{e^{-m}\, m^{x}}$$

$$= \frac{m}{x+1}$$

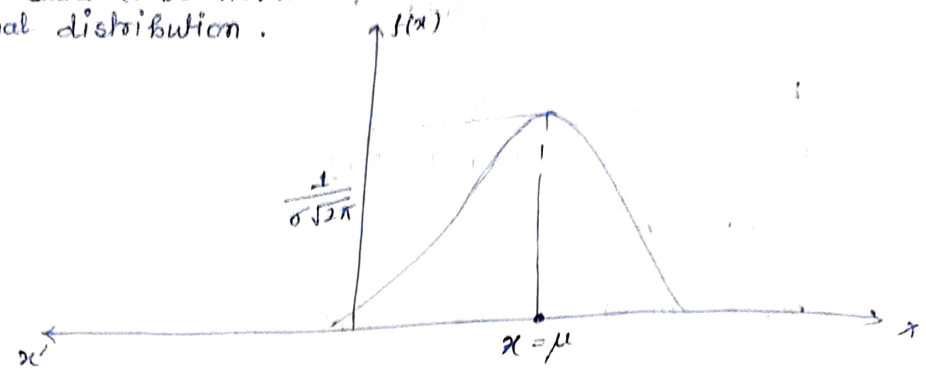
2 for P.D ; P.T the mean distribution about is $2/e$ if $m=1$

If a random variable X has the p.d.f given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$-\infty < x < \infty$
$-\infty < M < \infty$
$\sigma > 0$

where, M and σ are two constant, X is said to be normal variable and the distribution is called normal distribution.



If $\mu = 0$, it represents the dirac delta function.

properties of Normal distribution :—

① $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

② Mean of Normal distribution is μ and varience is σ^2.

③ The mode of the distribution $x = \mu$

④ The median of the " is μ.

⑤ The mean deviation about mean is, $\frac{4}{5}\sigma$.

Q Find the mean, varience & S.D of Gaussian distribution.

Soln pdf of normal distribution,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Mean, $E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma y + \mu)\, e^{-y^2} \sqrt{2}\sigma\, dy$$

let,

$\frac{(x-\mu)^2}{2\sigma^2} = y^2$

$\frac{x-\mu}{\sqrt{2}\sigma} = y$

$x = \sqrt{2}\sigma y + \mu$

$dx = \sqrt{2}\sigma\, dy$

$$= \frac{1}{\sigma\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} 2\sigma^2 y e^{-y^2} dy + \int_{-\infty}^{\infty} \mu\sqrt{2}\sigma e^{-y^2} dy \right]$$

$$= \frac{\mu\sqrt{2}\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy + \frac{2\sigma^2}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-y^2} dy$$

$$= \frac{2\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y^2} dy + \frac{2\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-y^2} dy$$

odd function

$y^2 = P \Rightarrow y = \sqrt{P}$

$dy = \frac{1}{2} P^{\frac{1}{2}-1} dP$

$= \frac{1}{2} P^{-1/2} dP$

$$= \frac{2\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-P} \frac{1}{2} P^{-1/2} dP$$

$$= \frac{\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-P} P^{1\bar{+}\frac{1}{2}-1} dP$$

$$= \frac{\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-P} P^{\frac{1}{2}-1} dP$$

$= \mu$ AND

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma y)^2 e^{-y^2} \sqrt{2}\sigma\, dy$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu^2 e^{-y^2} \sqrt{2}\sigma\, dy + \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\cdot\mu\cdot\sqrt{2}\sigma y e^{-y^2} \sqrt{2}\sigma\, dy$$
$$+ \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sigma^2 y^2 e^{-y^2} \sqrt{2}\sigma\, dy$$

$$= \frac{\mu^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy + \frac{2\sqrt{2}\sigma\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} y e^{-y^2} dy + \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2} dy$$

$$= \frac{\mu^2}{\sqrt{\pi}} \sqrt{\pi} + \frac{2\sigma^2 \cdot 2}{\sqrt{\pi}} \int_{0}^{\infty} P e^{-P} \frac{1}{2} P^{-1/2} dP$$

$$= \frac{\mu^2}{\sqrt{\pi}}\sqrt{\pi} + \frac{4\sigma^2}{2\sqrt{\pi}}\int_0^\infty e^{-P} P^{-\frac{1}{2}+1+1-1}\, dP$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}}\int_0^\infty e^{-P} P^{\frac{3}{2}-1}\, dP$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}}\left\lceil\frac{3}{2}\right.$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}}\frac{1}{2}\left\lceil\frac{1}{2}\right.$$

$$= \mu^2 + \sigma^2$$

$$\text{Variance} = E(x^2) - \{E(x)\}^2$$

$$= \mu^2 + \sigma^2 - \mu^2$$

$$= \sigma^2$$

$$S.D = \sqrt{\text{Variance}}$$

$$= \sigma,$$

Find the mode of Gaussian Distribution function.

$\Longrightarrow$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

To find the mode of the distribution function, we have to maximize $f(x)$.

$$\therefore f'(x) = 0$$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{d}{dx}\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] = 0$$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} (-)\frac{1}{2\sigma^2}\, 2(x-\mu)^{2-1}(1) = 0$$

$$\Rightarrow -\frac{(x-\mu)}{\sigma^3\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} = 0$$

$$(x-\mu)\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} = 0$$

$$\therefore \boxed{x = \mu}$$

$f''(x) = 0$

$$= \left[(1)\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} + (x-\mu)\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left[-\frac{1}{2\sigma^2} 2(x-\mu)\cdot 1 \right] \right] \frac{-1}{\sigma^3\sqrt{2\pi}}$$

$$= \left\{ e^{-\frac{(x-\mu)^2}{2\sigma^2}} - \frac{(x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma^2} \right\} \left(-\frac{1}{\sigma^3\sqrt{2\pi}} \right)$$

$$= e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left[1 - \frac{(x-\mu)^2}{\sigma^2} \right] \left(-\frac{1}{\sigma^3\sqrt{2\pi}} \right)$$

$$= e^{0} \cdot \left[\frac{(x-\mu)^2}{\sigma^2} - 1 \right] \left[\frac{1}{\sigma^3\sqrt{2\pi}} \right]$$

$$= \frac{-1}{\sigma^3\sqrt{2\pi}} < 0$$

So, $f(x)$ is maximum at $x = \mu$

Mode of normal or Gaussian distribution is $x = \mu$. Ans.

Q Find the mean deviation for Gaussian distribution func.

Soln: Mean deviation;

$$E(x - \bar{x}) = \int_{-\infty}^{\infty} |x-\mu|\, f(x)\, dx \qquad [\bar{x} = \mu]$$

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_{0}^{\infty} |x-\mu|\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

$\frac{(x-\mu)^2}{2\sigma^2} = p$

$(x-\mu)^2 = 2p\sigma^2$

$(x-\mu) = \sigma\sqrt{2p}$

$x = \sigma\sqrt{2p} + \mu$

$dx = \sigma\sqrt{2}\, \frac{1}{2} p^{\frac{1}{2}-1} dp$

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_{0}^{\infty} (\sigma\sqrt{2p})\, e^{-p}\, \frac{\sigma\sqrt{2}}{2} p^{\frac{1}{2}-1}\, dp$$

$$= \frac{\sigma\sqrt{2}\cdot\sigma\sqrt{2}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-p}\, p^{\frac{1}{2}+\frac{1}{2}-1}\, dp$$

$$= \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-p}\, p^{\frac{2}{2}-1}\, dp$$

$$= \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \quad \text{Ans}$$

(6) for normal distribution prove that,

(i) $P_{2n} = \sigma^2 (2n-1) P_{2n-2}$

(ii) $P_{2n} = \sigma^{2n} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)$

(iii) $P_{2n-1} = 0$

Soln : (i) Pdf of normal distribution,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Now, $P_{2n} = E\left\{(x-\bar{x})^{2n}\right\}$

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty (x-\mu)^{2n} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{2\sqrt{2}\sigma}{2\sigma\sqrt{2\pi}} \int_0^\infty 2\sigma^2 P \left(\sqrt{2P}\sigma\right)^{2n} \cdot e^{-P} P^{\frac{1}{2}-1} dP$$

$\frac{(x-\mu)^2}{2\sigma^2} = P$

$(x-\mu)^2 = 2\sigma^2 P$

$x - \mu = \sqrt{2P}\,\sigma$

$x = \sqrt{2P}\,\sigma + \mu$

$dx = \sqrt{2}\,\sigma\, \frac{1}{2} P^{\frac{1}{2}-1} dP$

$(x-\mu)^{2n} = \left(\sqrt{2P}\,\sigma\right)^{2n}$

$$= \frac{1 (\sqrt{2}\sigma)^{2n}}{\sqrt{\pi}} \int_0^\infty e^{-P} P^{\frac{2n}{2}} P^{\frac{1}{2}-1} dP \quad \text{—— (1)}$$

$$= \frac{(\sqrt{2}\sigma)^{2n}}{\sqrt{\pi}} \int_0^\infty e^{-P} P^{\frac{1}{2}+2n-1} dP$$

$\frac{1}{2}(2n-1)$

$n - \frac{1}{2}$

$$= \frac{(\sqrt{2}\sigma)^{2n}}{\sqrt{\pi}} \int_0^\infty e^{-P} P^{\left(\frac{1+2n}{2}\right)-1} dP$$

$$= \frac{(\sqrt{2}\sigma)^{2n}}{\sqrt{\pi}} \int_0^\infty e^{-P} \cdot \left\lceil \frac{1+2n}{2} \right\rceil \quad \text{—— (1)}$$

$\lceil n+1 = n\lceil n$

$$= \frac{(\sqrt{2}\sigma)^{2n}}{\sqrt{\pi}} \left\lceil \frac{1}{2} + n + 1 - 1 \right\rceil$$

$$= \frac{(\sqrt{2}\sigma)^{2n}}{\sqrt{\pi}} \left\lceil \frac{-1}{2} + n + 1 \right\rceil$$

$$= \frac{(\sqrt{2}\sigma)^{2n}}{\sqrt{\pi}} \left(n - \frac{1}{2}\right) \left\lceil \left(n - \frac{1}{2}\right) + 1 - 1 \right\rceil$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \left\lceil n - \frac{3}{2} \right\rceil$$

$$= \frac{2^n}{\sqrt{\pi}} \sigma^{2n} \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \cdots \frac{1}{2} \left\lceil \frac{1}{2} \right\rceil$$

$$= 2^n \sigma^{2n} \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right) \cdots \frac{1}{2}$$

$$= 2^n \sigma^{2n} \frac{(2n-1)(2n-3) \cdots 1}{2^n}$$

$$= \sigma^{2n} (2n-1)(2n-3) \cdots 1$$

$$\boxed{P_{2n} = \sigma^{2n} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)(2n-1)} \quad \text{---(III)}$$

If $2n$ is replaced by $(2n-2)$ in (III)

$$P_{2n-2} = \sigma^{2n-2} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-2-1)$$

$$P_{2n-2} = \sigma^{2n-2} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3) \quad \text{--- (IV)}$$

$$\frac{P_{2n}}{P_{2n-2}} = \frac{\sigma^{2n}}{\sigma^{2n-2}} (2n-1)$$

$$\boxed{P_{2n} = \sigma^2 (2n-1) P_{2n-2}}$$

replacing $(2n)$ by $(2n-1)$ in eqn (I)

$$P_{2n-1} = \frac{(\sqrt{2}\sigma)^{2n-1}}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^{2n-1} e^{-p^2} dp$$

$$P_{2n-1} = 0 \qquad \left[\text{As } f(p) = p^{2n-1} e^{-p^2} \text{ is an odd fun}^{c}\right]$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma p)^{2n} e^{-p^2} \sqrt{2}\sigma \, dp$$

$$= \frac{(\sqrt{2}\sigma)^{2n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^{2n+1} e^{-p^2} dp$$

Let, $\frac{(x-\mu)^2}{2\sigma^2} = p^2$

$$= 2^n \sigma^{2n} \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right) \cdots \frac{1}{2}$$

$$= 2^n \sigma^{2n} \frac{(2n-1)(2n-3)\cdots 1}{2^n}$$

$$= \sigma^{2n} (2n-1)(2n-3) \cdots 1$$

$$\boxed{P_{2n} = \sigma^{2n} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)(2n-1)} \quad \text{(III)}$$

If $2n$ is replaced by $(2n-2)$ in (III)

$$P_{2n-2} = \sigma^{2n-2} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-2-1)$$

$$P_{2n-2} = \sigma^{2n-2} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3) \quad \text{(IV)}$$

$$\frac{P_{2n}}{P_{2n-2}} = \frac{\sigma^{2n}}{\sigma^{2n-2}} (2n-1)$$

$$\boxed{P_{2n} = \sigma^2 (2n-1) P_{2n-2}}$$

replacing $(2n)$ by $(2n-1)$ in eqn (I)

$$P_{2n-1} = \frac{(\sqrt{2}\sigma)^{2n-1}}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^{2n-1} e^{-p^2} dp$$

$$P_{2n-1} = 0 \qquad [\because f(p) = p^{2n-1} e^{-p^2} \text{ is an odd fun}^{c}]$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma p)^{2n} e^{-p^2} \sqrt{2}\sigma \, dp$$

$$= \frac{(\sqrt{2}\sigma)^{2n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^{2n} e^{-p^2} dp$$

Let, $\frac{(x-\mu)^2}{2\sigma^2} = p^2$